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Dynamic Optimization Algorithm of Constrained Motion

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The constrained motion requires the determination of constraint force acting on unconstrained systems for satisfying given constraints. Most of the methods to decide the force depend on numerical approaches such that the Lagrange multiplier method, and the other methods need vector analysis or complicated intermediate process. In 1992, Udwadia and Kalaba presented the generalized inverse method to describe the constrained motion as well as to calculate the constraint force. The generalized inverse method has the advantages which do not require any linearization process for the control of nonlinear systems and can explicitly describe the motion of holonomically and/or nonholonomically constrained systems. In this paper, an explicit equation to describe the constrained motion is derived by minimizing the performance index, which is a function of constraint force vector, with respect to the constraint force. At this time, it is shown that the positive-definite weighting matrix in the performance index must be the inverse of mass matrix on the basis of the Gauss's principle and the derived differential equation coincides with the generalized inverse method. The effectiveness of this method is illustrated by means of two numerical applications.

Key Words : Constraints, Constraint Forces, Control, Dynamic Optimization, Generalized Inverse Method

1. Introduction

The motion of particles is sometimes restricted by desired trajectories. At this time, the constraint force acts on the particles for sustaining the motion along the trajectories. Thus, it is necessary to determine the constraint force to satisfy the given constraints for the description of a constrained motion. There have been many attempts to describe the constrained motion, and most of the methods to describe the constrained motion depend on numerical approaches like the Lagrange multiplier method (Gear, et al., 1985, Ascher and Petzold, 1993), or require complicated intermediate processs like vector analysis or elimination of configuration space as many as the number of constraints (Hemami and Weimer, 1981, Kane, 1983). In 1992, Udwadia and Kalaba (Udwadia and Kalaba, 1992) presented the generalized inverse method to determine the constrained motion as well as the constraint force. The generalized inverse method has the advantages which do not require any linearization pro-

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cess for the control of nonlinear systems and can explicitly describe the motion of holonomically and/or nonholonomically constrained systems.

The validity of the generalized inverse method has been examined by comparing with other approaches and applying to constrained mechanical and structural systems. For its verification, an explicit equation to describe the constrained motion is derived by minimizing a performance index with respect to the constraint force. It is proved that the unknown positive-definite weighting matrix in the performance index must be the inverse of mass matrix by comparing with Gauss's principle (Gauss, 1829) and the constraint force is defined as the minimum force of all forces to satisfy the given constraints. The derived differential equation of constrained systems coincides with the generalized inverse method. The effectiveness of this method is illustrated by means of two numerical applications.

2. Constraint force

The matrix equation of motion of a system modeled by an n-degree-of-freedom lumped mass-spring-dashpot system to include the effect of control force can be written as

$$\mathrm{M}\ddot{\mathrm{x}}(t)+\mathrm{C}\dot{\mathrm{x}}(t)+\mathrm{K}\mathrm{x}(t)=\mathrm{D}\mathrm{u}(t)+\mathrm{E}\mathrm{f}(t) \quad (1)$$

where M, C, and K are, respectively, the $n\times n$ mass, damping, and stiffness matrices, $\mathrm{x}(t)$ is the n-dimensional displacement vector, $\mathrm{f}(t)$ is an r-vector representing the applied load or external excitation, and $\mathrm{u}(t)$ is the m-dimensional control force vector. The $n\times m$ matrix D and $n\times r$ matrix E are location matrices which define locations of the control force and the excitation, respectively.

The state-space vector representation of Eq. (1) can be written by

$$\dot{\mathrm{z}}(t)=\mathrm{A}\mathrm{z}(t)+\mathrm{B}\mathrm{u}(t)+\mathrm{W}\mathrm{f}(t) \quad (2)$$

where $\mathrm{z}(t)=[\mathrm{x}(t) \quad \dot{\mathrm{x}}(t)]^{\mathrm{T}}$ is the $2n$-dimensional state vector,

$$\mathrm{A}=\begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{M}^{-1}\mathrm{K} & -\mathrm{M}^{-1}\mathrm{C}\end{bmatrix}_{2n\times 2n},$$

$$\mathrm{B}=\begin{bmatrix} 0 \\ \mathrm{M}^{-1}\mathrm{D}\end{bmatrix}_{2n\times m}, \text{ and } \mathrm{W}=\begin{bmatrix} \mathrm{I} \\ \mathrm{M}^{-1}\mathrm{E}\end{bmatrix}_{2n\times r}.$$

Assuming that a system is unconstrained, quadratic optimal control can be found by determining an optimal control vector $\mathrm{u}(t)$ so as to minimize the performance index

$$J=\int_0^\infty (\mathrm{x}^{\mathrm{T}}\mathrm{Q}\mathrm{x}+\mathrm{u}^{\mathrm{T}}\mathrm{R}\mathrm{u})\,dt, \quad (3)$$

where Q is a positive-semidefinite Hermitian or real symmetric matrix and R is a positive-definite Hermitian or real symmetric matrix. But this control algorithm can not apply to the constrained systems because $\mathrm{u}(t)$ is the unconstrained control vector. The control algorithm of constrained systems can be derived as follows.

To describe the motion of constrained systems, we assume that the n-degree-of-freedom system is constrained by the m consistent constraints

$$\phi_i(\mathrm{x}, \dot{\mathrm{x}}, t)=0, \qquad i=1, 2, \cdots, m \quad (4)$$

of which $m<n$. The state variables must satisfy the constraint sets due to the constraint force provided by Nature at all times. The general equation of motion at time t of constrained systems can be expressed as

$$\mathrm{M}\ddot{\mathrm{x}}=\mathrm{F}(\mathrm{x}, \dot{\mathrm{x}}, t)+\mathrm{v}(\mathrm{x}, \dot{\mathrm{x}}, t), \quad (5)$$

where $\mathrm{F}(\mathrm{x}, \dot{\mathrm{x}}, t)=-\mathrm{C}\dot{\mathrm{x}}(t)-\mathrm{K}\mathrm{x}(t)+\mathrm{E}\mathrm{f}(t)$, and $\mathrm{v}(\mathrm{x}, \dot{\mathrm{x}}, t)$ is the constraint force vector.

The first term of the performance index (3) is excluded because the state variables are prescribed and governed by the values to satisfy the given constraints. And, the constraint force vector $\mathrm{v}(t)$ for the constrained motion replaces the control force vector $\mathrm{u}(t)$ for the optimal control of unconstrained systems. Consequently, the constraint force vector $\mathrm{v}(t)$ in the performance index (3) can be derived by minimizing the performance index

$$J=\int_0^\infty \mathrm{v}^{\mathrm{T}}\mathrm{R}\mathrm{v}\,dt, \qquad \mathrm{v}\neq 0 \quad (6)$$

where R is also a positive-definite Hermitian or real symmetric matrix.

Under the assumption that the constraint equations are sufficiently smooth functions to be differentiated with respect to time t, the proper differentiation of Eq. (4) with respect to time t

leads to the linear set of equations

$$S(x, \dot{x}, t)\ddot{x} = b(x, \dot{x}, t), \tag{7}$$

where S is an $m \times n$ matrix, and b is an $m \times 1$ vector. Substituting $\ddot{x} = M^{-1}(F+v)$ from Eq. (5) into Eq. (7), we obtain the equation

$$S[M^{-1}F + M^{-1}v] = b \tag{8a}$$

or

$$(SM^{-1}R^{-1/2})R^{1/2}v = b - Sa, \tag{8b}$$

where a is the acceleration vector of the unconstrained system. Letting $SM^{-1}R^{-1/2} = H$ in Eq. (8b) and solving it, it follows that

$$R^{1/2}v = H^{+}(b - Sa) + [I - H^{+}H]y, \tag{9}$$

where y is an arbitrary vector and '+' indicates the generalized inverse.

The minimization of the performance index (6) with respect to v indicates $R^{1/2}v = 0$, and the arbitrary vector y of Eq. (9) can be expressed as

$$\begin{aligned} y = &[I - H^{+}H]H^{+}(b - Sa) \\ &+ [I - [I - H^{+}H]^{+}[I - H^{+}H]]z \end{aligned} \tag{10}$$

where z is another arbitrary vector. Using Eq. (10) into Eq. (9) with $[I - H^{+}H]^{+} = I - H^{+}H$, $H^{+}HH^{+} = H^{+}$, we can obtain the arbitrary vector y as

$$y = H^{+}Hz. \tag{11}$$

The substitution of Eq. (11) into Eq. (9) yields the constraint force vector

$$v = R^{-1/2}(SM^{-1}R^{-1/2})^{+}(b - Sa). \tag{12}$$

Using Eq. (12) into Eq. (5), we can obtain an explicit equation of motion for constrained systems. However, the constraint force depends on the weighting matrix R as shown in Eq. (12). It is necessary to decide the weighting matrix R so that the constraint force does not violate the constraint conditions.

From the Gauss's principle the acceleration, $\ddot{x}(t)$ which minimize the Gaussian function, G, defined by

$$G = [\ddot{x} - a]^{T}M[\ddot{x} - a], \tag{13}$$

In actual acceleration provided by nature. Here, the $n \times n$ mass matrix M is symmetric and positive definite. Using Eq. (5) into Eq. (13), we can write the Gaussian function in terms of the constraint force vector as follows:

$$G = v^{T}M^{-1}v. \tag{14}$$

Through the comparison of Eqs. (6) and (14), it is apparent that the unknown weighting matrix R takes the inverse of mass matrix M^{-1}. The final equation of the constraint force can be expressed as

$$v = M^{1/2}(SM^{-1/2})^{+}(b - Sa). \tag{15}$$

The constrained motion is sustained by the action of the constraint force determined by Eq. (15). It can be observed that the constraint force is the minimum value of all forces to satisfy the constraints on the basis of the Gauss's principle. In additudn, it is indicated that the Gauss's principle provides another meaning that the constraint force such that the Gaussian function given by Eq. (13) is minimized over all the constraint forces, and it satisfies the constraints. The constraint force presented by the generalized inverse method is determined by the same form as Eq. (15) in spite of another interpretation of the Gauss's principle. The validity of this method is established from the following two applications.

3. Application 1

To show the effectiveness of this method, we considered the coupled Duffing's oscillator shown in Fig. 1, which is subjected to a constraint

$$x_1(t) - x_2(t) = \beta e^{-\alpha t}\sin(\omega t). \tag{16}$$

The two nonlinear springs s_1 and s_2 exert forces representad by

$$f_i = k_i u_i + k_i^{nl} u_i^3, \qquad i = 1, 2 \tag{17}$$

where u_i denotes the extension of the ith spring and k_i^{nl} indicates that the spring force has a cubic nonlinearity. The equation of motion of the unconstrained system given by Fig. 1 may be written

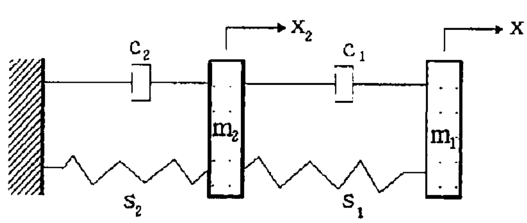

Fig. 1 Coupled Duffing's oscillator

as

$$F = M\ddot{x} = -[Kx + C\dot{x} + g^{nl}] \tag{18a}$$

or

$$a(t) = [a_1(t) \quad a_2(t)]^T = -M^{-1}[Kx + C\dot{x} + g^{nl}] \tag{18b}$$

where $M = diag[m_1, m_2]$,

$$K = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix},$$

$$C = \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 \end{bmatrix}, \text{ and}$$

$$G^{nl} = \begin{bmatrix} k_1^{nl}(x_1 - x_3)^3 \\ k_2^{nl}x_2^3 - k_1^{nl}(x_1 - x_2)^3 \end{bmatrix}.$$

Differentiating Eq. (16) twice, we get the constraint equation

$$\ddot{x}_1 - \ddot{x}_2 = -\beta e^{-\alpha t}[\omega^2 \sin \omega t + 2\omega\alpha \cos \omega t - \alpha^2 \sin \omega t] = b(t). \tag{19}$$

Hence, the matrix $S = [1 \quad -1]$ and $b(t)$ is a scalar. The constraint force from Eq. (12) can be written as

$$v(t) = [v_1 \; v_2]^T = R^{-1/2}\left([1 \; -1]\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}^{-1} R^{-1/2}\right)^{+} [b(t) - a_1(t) + a_2(t)] \tag{20}$$

The parameter values describing the system are: $m_1 = 2$, $m_2 = 1$, $k_1 = 10$, $k_2 = 12$, $k_1^{nl} = 1$, $k_2^{nl} = 2$, $c_1 = 0.1$, $c_2 = 0.15$, $\beta = 1$, $\omega = 2\pi$, $\alpha = 1$. The initial conditions are:

$$x_1(0) = x_2(0) = 1, \quad \dot{x}_1(0) = \beta\omega + \dot{x}_2(0), \quad \dot{x}(0) = 2 \tag{21}$$

As shown in Eq. (20), the constraint force and the constrained response depend on the weighting matrix. Figure 2 shows the numerical results according to three different weighting matrices; $R1 = M^{-1}$, $R2 = diag[1 \quad 1]$, $R3 = diag[0.5 \quad 0.3]$. As shown in these figures, both the minimum response and constraint force occur at $R1 = M^{-1}$. Figure 2(d) shows the errors in the satisfaction of the constraint (16), and the displacement responses satisfy the constraint equa-

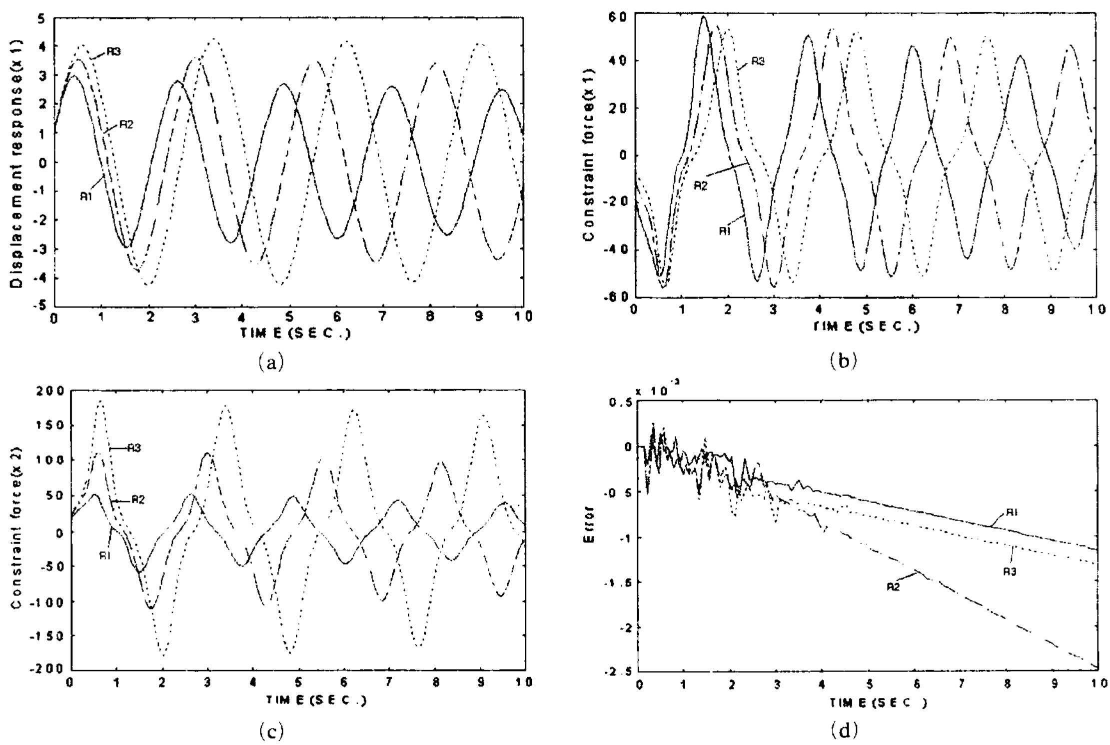

Fig. 2 Displacement responses and constraint forces according to weighting matrix; (a) displacement responses, (b) constraint force in the x_1-direction, (c) constraint force in the x_2-direction, (d) error in the satisfaction of the constraint eq. (16)

tion (16) regardless of the selected weighting matrix. This is due to many constraint forces to satisfy the constraint as indicated by the arbitrary vector y in Eq. (9). Based on the Gauss's principle and the numerical results, the weighting matrix should be the inverse of mass matrix. Therefore, the constraint force can be physically defined as the minimum force of all forces to satisfy given constraints.

4. Application 2

A single story structure installing a double pendulum indicated by Fig. 3 has mass m_p, stiffness k_p, damping coefficient c_p. The double pendulum is connected by a hinge at the top of floor, and a roller with stiffness k and damping

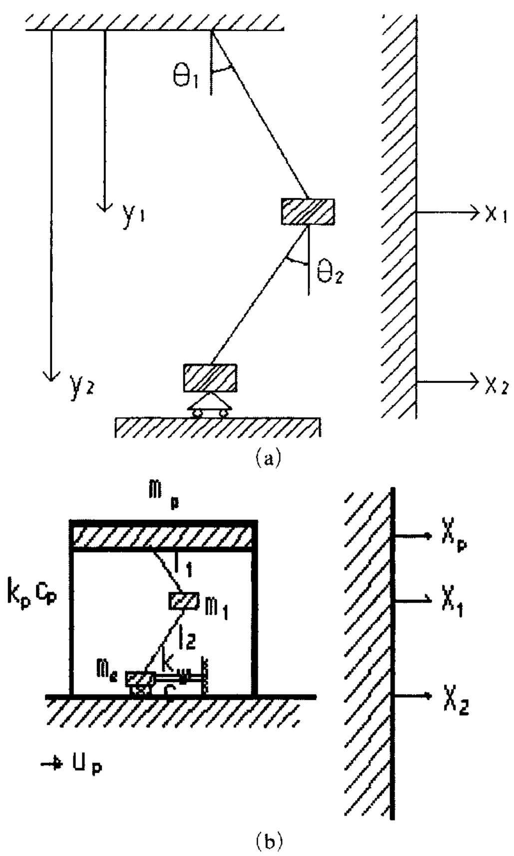

Fig. 3 A double pendulum at the top floor of structure; (a) A double pendulum, (b) constrained double pendulum

coefficient c at the bottom of floor. The pendulum has two masses m_1 and m_2, and two weightless lengths l_1 and l_2. This system can be described by the Lagrangian coordinates $[x_p\ \theta_1\ \theta_2]^T$ or Cartesian coordinates $[x_p\ x_1\ y_1\ x_2\ y_2]^T$. The unconstrained equations of motion of the given system by the Cartesian coordinate system are written as

$$m_p\ddot{x}_p+(k_p+k)x_p+(c_p+c)\dot{x}_p-kx_2-c\dot{x}_2=-m_p\ddot{u}_g \tag{22a}$$

$$m_1\ddot{x}_1=-m_1\ddot{u}_g \tag{22b}$$

$$m_1\ddot{y}_1=-m_1 g \tag{22c}$$

$$m_2\ddot{x}_2-kx_p+kx_2-c\dot{x}_p+c\dot{x}_2=-m_2\ddot{u}_g \tag{22d}$$

$$m_2\ddot{y}_2=-m_2 g \tag{22e}$$

where g is the acceleration of gravity and $\ddot{u}_g$ is the ground acceleration.

The system has three constraints, two of which are expressed as

$$(x_1-x_p)^2+y_1^2=l_1^2 \tag{23a}$$

and

$$(x_2-x_1)^2+(y_2-y_1^2)=l_2^2 \tag{23b}$$

which indicate the relation of the Lagrangian and Cartesian coordinates. And the other constraint, which the height of floor is always equal to the one of the pendulum, is given by

$$y_2=h \tag{23c}$$

where h is the floor height.

Differentiating twice the constraint equation sets (23) with respect to time t, we obtain the relation

$$\begin{bmatrix} x_p-x_1 & x_1-x_p & y_1 & 0 & 0 \\ 0 & x_1-x_2 & y_1-y_2 & x_2-x_1 & y_2-y_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_p \\ \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{x}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} -(\dot{x}_1-\dot{x}_p)^2-\dot{y}_1^2 \\ -(\dot{x}_2-\dot{x}_1)^2-(\dot{y}_2-\dot{y}_1)^2 \\ 0 \end{bmatrix} \tag{24}$$

Using Eqs. (22) and (24) in Eqs. (5) and (15), we can explicitly determine the constrained equation of motion.

The above systems are described by the values

$$h=3.0,\ m_p=3.0,\ m_1=0.3,\ m_2=0.03,\ l_1=2.4,\ l_2=0.9,\ k_p=400,\ k=15,\ c_p=c=1.386 \tag{25}$$

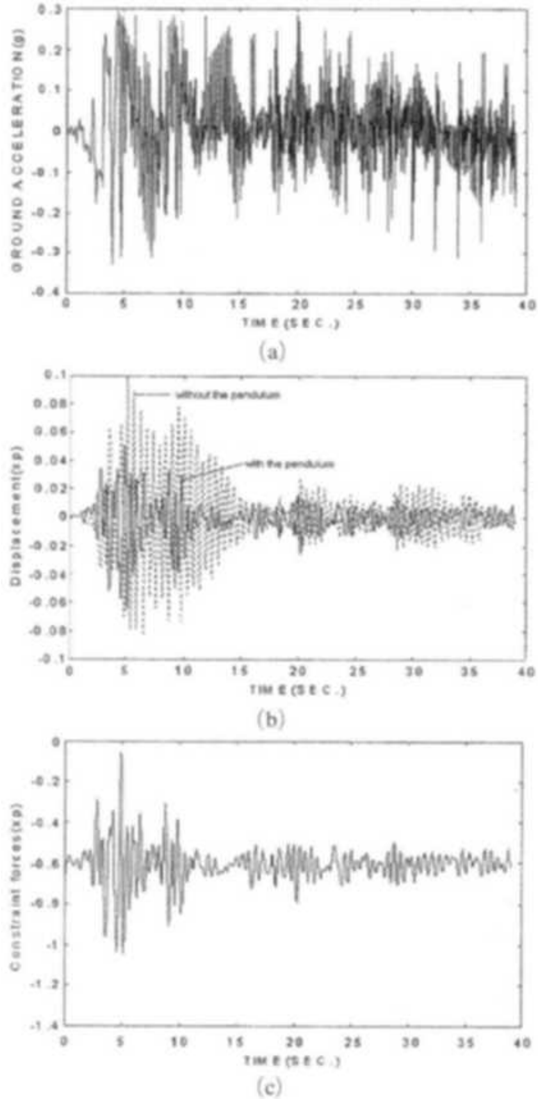

Fig. 4 Dynamic responses with or without the pendulum ; (a) North-south components of El-Centro earthquake in 1940, (b) displacement responses with or without the pendulum, (c) constraint forces acting in the x_p direction

Also, we choose the initial conditions

$$
\begin{aligned}
&x_1(0)=0.5953,\ y_1(0)=2.325,\ y_2(0)=3.0 \\
&x_p(0)=x_2(0)=\dot{x}_p(0)=\dot{x}_1(0)=\dot{y}_1(0) \\
&\quad=\dot{x}_2(0)=\dot{y}_2(0)=0
\end{aligned} \tag{26}
$$

Assume that the structure was excited by the north-south components of El Centro earthquake in 1940 shown by Fig. 4(a). In Fig. 4(b), we compared the structural responses with and without the double pendulum. As shown in the figure, the installation of the pendulum changes the dynamic characteristics and yields the reduced responses by the acting constraint force. Figure 4 (c) shows the control force acting in the x_p direction calculated by Eq. (15). The force can be interpreted as the minimum force of all forces to satisfy the given constraints. This application also exhibits that the constrained response and constraint force can be simply and explicitly decided. From the applications, it is convinced that the proposed method can be simply applied to the control field of constrained structural or mechanical systems.

5. Conclusions

The exact description of constrained motion depends on the explicit determination of constraint force provided by Nature in order to satisfy constraints, which restrict the motion of systems. There have been many methods to describe the constrained motion, and most of them depend on numerical approaches such that the Lagrange multiplier method. Thus, this paper derived an explicit equation to describe the constrained motion as well as to calculate the constraint force by minimizing a performance index, which is a function of constraint force vector, with respect to the constraint force. The comparison with Gauss's principle yielded that the weighting matrix in the performance index must be the inverse of mass matrix. The derived differential equation coincided with the generalized inverse method and Gauss's principle can be interpreted as another meaning that Nature chooses the constraint force, which is the minimum force of all forces to satisfy constraints. Two applications illustrated that the generalized inverse method can be applied to various control fields of mechanical and structural systems.

References

Ascher, U. M. and Petzold, L. R., 1993, "Stability of Computational Methods for Constrained

Dynamic System," *SIAM Journal on Scientific Computing*, Vol. 14, No. 1, pp. 95~120.

Gear, C. W., Leimkuhler, B. and Gupta, G. K., 1985, "Automatic Integration of Euler-Lagrange Equations with Constraints," *Journal of Computational and Applied Mathematics*, pp. 77~90.

Gauss, C. F., 1829, "Uber ein Neues Allgemeines Grundgesetz der Mechanik," *Journal Fur Die Reine Und Angewandte Mathematik*, Vol. 4, pp. 232~235.

Hemami, H. and Weimer, F. C., 1981, "Modeling of Nonholonomic Dynamic Systems with Applications," *ASME, Journal of Applied Mechanics*, Vol. 48, pp. 177~182.

Kane, T. R., 1983, "Formulation of Dynamical Equations of Motion," *American Journal of Physics*, Vol. 51, pp. 974~977.

Udwadia, F. E. and Kalaba, R. E., 1992, "A New Perspective on Constrained Motion," *Proceedings of the Royal Society of London*, Vol. 439, pp. 407~410.